

Nilpotent Symmetries and Curci-Ferrari Type Restrictions in $2D$ Non-Abelian Gauge Theory: Superfield Approach

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Abstract: We derive the off-shell nilpotent symmetries of the two (1+1)-dimensional ($2D$) non-Abelian 1-form gauge theory by using the theoretical techniques of the geometrical superfield approach to Becchi-Rouet-Stora-Tyutin (BRST) formalism. For this purpose, we exploit the augmented version of superfield approach (AVSA) and derive theoretically useful nilpotent (anti-)BRST, (anti-)co-BRST symmetries and Curci-Ferrari (CF) type restrictions for the self-interacting $2D$ non-Abelian 1-form gauge theory (where there is *no* interaction with matter fields). The derivation of the (anti-)co-BRST symmetries and *all* possible CF-type restrictions are completely *novel* results within the framework of AVSA to BRST formalism where the *ordinary* $2D$ non-Abelian theory is generalized onto an appropriately chosen (2, 2)-dimensional supermanifold. The latter is parameterized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ where x^μ (with $\mu = 0, 1$) are the bosonic coordinates and a pair of Grassmannian variables $(\theta, \bar{\theta})$ obey the relationships: $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} + \bar{\theta}\theta = 0$.

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1 Introduction

The principles of *local* gauge symmetries (and their consequences) are at the heart of the precise theoretical description of three (out of four) fundamental interactions of nature [1]. The gauge theories, based on the *above* local symmetries, are quantized covariantly and consistently within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism where the local gauge symmetries of the original *classical* gauge theories are traded with the quantum gauge [i.e. (anti-)BRST] symmetries (at the *quantum* level). For a given local gauge symmetry, there exist *two* quantum gauge symmetries (within the framework of BRST formalism) which are christened as the BRST and anti-BRST symmetries. The nilpotency and absolute anticommutativity properties are the *two* decisive features of these symmetries which encompass in their folds the properties of “supersymmetry” and linear independence, respectively. In other words, the transformations generated by the (anti-)BRST symmetries are fermionic (i.e. supersymmetric-type) in nature and they have their own independent identities due to their absolute anticommutativity property.

The usual superfield approach (USFA) to BRST formalism [2-9] sheds light on the properties of nilpotency and absolute anticommutativity because the (anti-)BRST symmetries are identified with the *translational generators* along a pair of Grassmannian variables $(\theta, \bar{\theta})$ which characterize the $(D, 2)$ -dimensional supermanifold on which a given D -dimensional *ordinary* gauge theory is generalized. To be precise, the $(D, 2)$ -dimensional supermanifold is parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$ where the bosonic coordinates x^μ (with $\mu = 0, 1, 2 \dots D-1$) correspond to the ordinary D -dimensional space-time variables and the Grassmannian variables $(\theta, \bar{\theta})$ satisfy the standard relationships: $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$. In the above *identification* (and geometrical *interpretation*), the celebrated horizontality condition (HC) plays a key role which primarily leads to the derivation of (anti-)BRST transformations for the *gauge* fields and the corresponding *(anti-)ghost* fields of a given D -dimensional gauge theory (described within the framework of USFA to BRST formalism).

The above USFA has been systematically and consistently generalized so as to derive the (anti-)BRST symmetry transformations for the gauge, (anti-)ghost and *matter* fields *together* for an *interacting* gauge theory where there is a coupling between the gauge fields and matter fields [10-14]. In the above generalization, in addition to the HC, we invoke additional restrictions [i.e. gauge invariant restriction (GIRs)] which are consistent with the HC and there is an inter-relationship and inter-dependence between the HC and GIRs in such a manner that the geometrical *interpretation* of the (anti-)BRST symmetries (and corresponding conserved charges) remains intact. The generalized version of the superfield approach has been christened as the augmented version of the superfield approach (AVSA) to BRST formalism [10-14]. We have exploited the latter superfield approach (i.e. AVSA) to discuss the central theme of our present endeavor where we have derived the off-shell nilpotent and absolutely anticommuting (anti-)BRST as well as (anti-)co-BRST symmetries and *all* the possible Curci-Ferrari (CF) type restrictions that would, in general, be supported by the $2D$ non-Abelian gauge theory under consideration. In our present work, we have, however, utilized *only* a few of the total CF-type restrictions (supported by our $2D$ theory).

In our present investigation, we concisely mention the results of [5,6] where we discuss the strength of HC in the derivation of proper (anti-)BRST symmetries for the non-Abelian

theory (in any arbitrary dimension of spacetime). The novelty of our present work begins with the derivation of proper (anti-)co-BRST symmetry transformations where we exploit the virtues of AVSA to BRST formalism. In fact, we utilize the ideas of dual-HC (DHC) and dual-gauge invariant restrictions (DGIRs) for the complete derivation of proper (anti-)co-BRST symmetry transformations for *all* the fields of our theory. The highlights of our present investigation are, however, the derivation of CF-type restrictions using the AVSA to BRST formalism where the inputs from the results, obtained from the application of HC, DHC, GIRs, as well as DGIRs, are utilized *together*. We have been able to compute *all* possible CF-type restrictions from the original CF-condition $[B + \bar{B} + (C \times \bar{C}) = 0]$ by requiring the (anti-)BRST and (anti-)co-BRST invariance of it within the framework of AVSA to BRST formalism. In fact, we have exploited primarily the basic tenets of AVSA.

Our present investigation is inspired and influenced by the following key factors. First, the $2D$ non-Abelian 1-form gauge theory (without any interaction with matter fields) is the *only* non-Abelian 1-form gauge model where we have been able to demonstrate the existence of (anti-)dual BRST [i.e. (anti-)co-BRST] symmetry transformations. Thus, it is challenging for us to derive these (anti-)co-BRST symmetry transformations from the AVSA. Second, the insights and understanding gained from our present endeavor would be useful in obtaining the (anti-)co-BRST symmetry transformations for the higher p -form ($p = 2, 3, \dots$) gauge theories within the framework of AVSA. In this connection, we mention that, for the $4D$ Abelian 2-form and $6D$ Abelian 3-form gauge theories, we have already shown the existence of the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations *together* [15-17]. Finally, one of the key signatures of the BRST approach to the p -form ($p = 1, 2, 3, \dots$) gauge theories is the existence of the CF-type restrictions. Thus, it is a challenging problem for us to derive them within the framework of AVSA (particularly in the cases where the (anti-)BRST and (anti-)co-BRST symmetries exist *together*). We have derived *all* possible CF-type restrictions that could be supported by the $2D$ non-Abelian theory where the (anti-)BRST and (anti-)co-BRST symmetries *co-exist*. However, only *a few* of these have been actually used by us in the discussion of the symmetries of our $2D$ theory within the framework of BRST formalism.

The material of our present investigation is organized as follows. In Sec. 2, we discuss very concisely the (anti-)BRST and (anti-)co-BRST symmetries for the $2D$ non-Abelian 1-form gauge theory in the Lagrangian formulation to set-up the notations and conventions. Our Sec. 3 is devoted to a brief synopsis of HC so that our paper could be self-contained. The subject matter of Sec. 4 is the application of the AVSA to derive the (anti-)co-BRST symmetry transformations using the DHC and DGIR for the $2D$ non-Abelian 1-form gauge theory. Our Sec. 5 deals with the derivation of *all* possible CF-type restrictions that could be supported by our $2D$ theory by using AVSA to BRST formalism. Finally, we make some concluding remarks in Sec. 6 and point out a few future directions for further investigations.

In our Appendix A, we discuss some explicit computations which have been incorporated in the main body of the text of our present endeavor.

Notations and Convention: Throughout the whole body of our text, we use the notations $s_{(a)b}$ and $s_{(a)d}$ for the (anti-)BRST and (anti-)dual-BRST symmetry transformations. The covariant derivative $D_\mu C = \partial_\mu C + i(A_\mu \times C)$ is in the adjoint representation of the $SU(N)$ Lie algebraic space where the generators T^a (with $a = 1, 2, \dots, N^2 - 1$) obey the Lie algebra

$[T^a, T^b] = f^{abc} T^c$. The structure constants f^{abc} are chosen to be totally antisymmetric for the semi-simple Lie group $SU(N)$. We further adopt the notations $P \cdot Q = P^a Q^a$ and $(P \times Q)^a = f^{abc} P^b Q^c$ where the Latin indices $a, b, c, \dots = 1, 2, 3, \dots, N^2 - 1$ and (P^a, Q^a) are chosen to be non-null vectors in the Lie-algebraic space. We choose the background $2D$ flat metric $\eta_{\mu\nu}$ with signatures $(+1, -1)$ so that $A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu \equiv A_0 B_0 - A_i B_i$ where the Greek indices $\mu, \nu, \lambda, \dots = 0, 1$ stand for the spacetime directions and the Latin indices $i, j, k, \dots = 1$ correspond to the space direction *only*. In addition, the $2D$ Levi-Civita tensor $\varepsilon_{\mu\nu}$ has been chosen such that $\varepsilon_{01} = +1 = \varepsilon^{10}$ and $\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2!$, $\varepsilon_{\mu\nu} \varepsilon^{\nu\lambda} = \delta_\mu^\lambda$, etc.

2 Nilpotent Symmetries: Lagrangian Formulation

Let us begin with the following coupled (but equivalent) (anti-)BRST invariant Lagrangian densities [18] for the $2D$ non-Abelian 1-form $(A^{(1)} = dx^\mu A_\mu \cdot T)$ gauge theory in the Curci-Ferrari gauge (see, e.g [19,20] for details)

$$\begin{aligned}\mathcal{L}_B &= -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\ \mathcal{L}_{\bar{B}} &= -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C,\end{aligned}\quad (1)$$

where B and \bar{B} are the Nakanishi-Lautrup type auxiliary fields and anticommuting $C^a \bar{C}^a + \bar{C}^a C^a = 0$ (anti-)ghost fields $(\bar{C}^a) C^a$ are fermionic $[(C^a)^2 = (\bar{C}^a)^2 = 0]$ in nature. The 2-form $F^{(2)} = dA^{(1)} + i A^{(1)} \wedge A^{(1)} \equiv (dx^\mu \wedge dx^\nu / 2!) (F_{\mu\nu} \cdot T)$ defines the curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i (A_\mu \times A_\nu)$ which has *only* one existing independent component in $2D$, namely; $F_{01} = \partial_0 A_1 - \partial_1 A_0 + i (A_0 \times A_1) \equiv E$. Thus, for the case of $2D$ non-Abelian theory, we have the following coupled Lagrangian densities corresponding to (1), namely;

$$\begin{aligned}\mathcal{L}_B^{(2D)} &= \frac{1}{2} E \cdot E + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\ \mathcal{L}_{\bar{B}}^{(2D)} &= \frac{1}{2} E \cdot E - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C.\end{aligned}\quad (2)$$

The above Lagrangian densities are *equivalent* on the constrained hypersurface where $B \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C = -B \cdot (\partial_\mu A^\mu) - i D_\mu \bar{C} \cdot \partial^\mu C$. This equality leads to the CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$ (modulo a total spacetime derivative term). The following supersymmetric-type (anti-)BRST transformations $(s_{(a)b})$

$$\begin{aligned}s_b A_\mu &= D_\mu C, & s_b C &= -\frac{i}{2} (C \times C), & s_b \bar{C} &= i B, \\ s_b \bar{B} &= i (\bar{B} \times C), & s_b F_{\mu\nu} &= i (F_{\mu\nu} \times C), & s_b B &= 0, \\ s_{ab} A_\mu &= D_\mu \bar{C}, & s_{ab} \bar{C} &= -\frac{i}{2} (\bar{C} \times \bar{C}), & s_{ab} C &= i \bar{B}, \\ s_{ab} B &= i (B \times \bar{C}), & s_{ab} F_{\mu\nu} &= i (F_{\mu\nu} \times \bar{C}), & s_{ab} \bar{B} &= 0,\end{aligned}\quad (3)$$

are the *symmetry* transformations for the action integral $S = \int d^2x \mathcal{L}_B \equiv \int d^2x \mathcal{L}_{\bar{B}}$ because one observes that the following are true, namely;

$$s_b \mathcal{L}_B = \partial_\mu [B \cdot D^\mu C], \quad s_{ab} \mathcal{L}_{\bar{B}} = \partial_\mu [-\bar{B} \cdot D^\mu \bar{C}],$$

$$\begin{aligned}
s_{ab}\mathcal{L}_B &= \partial_\mu[-\{\bar{B} + (C \times \bar{C})\} \cdot \partial^\mu \bar{C}] + [B + \bar{B} + (C \times \bar{C})] \cdot D_\mu \partial^\mu \bar{C}, \\
s_b\mathcal{L}_{\bar{B}} &= \partial_\mu[\{B + (C \times \bar{C})\} \cdot \partial^\mu \bar{C}] - [B + \bar{B} + (C \times \bar{C})] \cdot D_\mu \partial^\mu C.
\end{aligned} \tag{4}$$

These relationships establish that the above symmetries are *true* on a constrained hypersurface, embedded in the $2D$ spacetime manifold, where the Curci-Ferrari (CF) condition $B + \bar{B} + (C \times \bar{C}) = 0$ is valid. It is elementary, at this stage, to note that we have $s_{ab}\mathcal{L}_B = \partial_\mu[B \cdot \partial^\mu \bar{C}]$ and $s_b\mathcal{L}_{\bar{B}} = -\partial_\mu[\bar{B} \cdot \partial^\mu C]$ due to the validity of CF-condition. Further, we note that the absolute anticommutativity ($s_b s_{ab} + s_{ab} s_b = 0$) property of the (anti-)BRST transformations $s_{(a)b}$ is true only when the (anti-)BRST invariant (i.e. $s_{(a)b}[B + \bar{B} + (C \times \bar{C})] = 0$) CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$ is imposed from outside. It is crucial to point out that the gauge-fixing and Faddeev-Popov ghost terms of the starting Lagrangian density (1) can be written as [19,20]

$$\begin{aligned}
\mathcal{L}_B &= -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + s_b s_{ab} \left(\frac{i}{2} A_\mu \cdot A_\mu - \frac{\xi}{2} \bar{C} \cdot C \right), \\
\mathcal{L}_{\bar{B}} &= -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - s_{ab} s_b \left(\frac{i}{2} A_\mu \cdot A_\mu - \frac{\xi}{2} \bar{C} \cdot C \right),
\end{aligned} \tag{5}$$

where the Curci-Ferrari gauge condition (cf. Eq. (1)) implies that we have chosen $\xi = 2$. Thus far, all our statements are true in any arbitrary dimension of spacetime. In other words, the (anti-)BRST symmetries (3) are *true* for any arbitrary non-Abelian 1-form gauge theory (when we discuss the theory within the framework of BRST formalism).

In addition to the above nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations $s_{(a)b}$, we *also* have a set of proper (i.e. nilpotent and absolutely anticommuting) (anti-)co-BRST symmetry transformations ($s_{(a)d}$) in our theory. These transformations, in the context of our $2D$ non-Abelian theory, are (see, e.g. [21]):

$$\begin{aligned}
s_{ad}A_\mu &= -\varepsilon_{\mu\nu}\partial^\nu C, & s_{ad}C &= 0, & s_{ad}\bar{C} &= i\mathcal{B}, & s_{ad}\mathcal{B} &= 0, \\
s_{ad}E &= D_\mu \partial^\mu C, & s_{ad}(\partial_\mu A^\mu) &= 0, & s_{ad}B &= 0, & s_{ad}\bar{B} &= 0, \\
s_d A_\mu &= -\varepsilon_{\mu\nu}\partial^\nu \bar{C}, & s_d \bar{C} &= 0, & s_d C &= -i\mathcal{B}, & s_d \mathcal{B} &= 0, \\
s_d E &= D_\mu \partial^\mu \bar{C}, & s_d(\partial_\mu A^\mu) &= 0, & s_d B &= 0, & s_d \bar{B} &= 0.
\end{aligned} \tag{6}$$

The above off-shell nilpotent (anti-)co-BRST transformations are the *symmetry* transformations of the following Lagrangian densities

$$\begin{aligned}
\mathcal{L}_B^{(2D)} &= \mathcal{B} \cdot E - \frac{1}{2} \mathcal{B} \cdot \mathcal{B} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i\partial_\mu \bar{C} \cdot D_\mu C, \\
\mathcal{L}_{\bar{B}}^{(2D)} &= \mathcal{B} \cdot E - \frac{1}{2} \mathcal{B} \cdot \mathcal{B} - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - iD_\mu \bar{C} \cdot \partial^\mu C,
\end{aligned} \tag{7}$$

where we have linearized the kinetic term ($\frac{1}{2}E \cdot E$) of the Lagrangian density (2) by introducing the auxiliary field \mathcal{B} . It is straightforward to check that the (anti-)co-BRST symmetry transformations ($s_{(a)d}$) are off-shell nilpotent ($s_{(a)d}^2 = 0$) of order two and they are absolutely anticommuting ($s_d s_{ad} + s_{ad} s_d = 0$). The former property ensures the fermionic (supersymmetric) nature of $s_{(a)d}$ and the latter property encodes the linear independence of s_d and s_{ad} . It can be readily checked that $s_d \mathcal{L}_B^{(2D)} = \partial_\mu [\mathcal{B} \cdot \partial^\mu \bar{C}]$ and $s_{ad} \mathcal{L}_{\bar{B}}^{(2D)} = \partial_\mu [\mathcal{B} \cdot \partial^\mu C]$. Hence the action integrals $S = \int d^2x \mathcal{L}_B^{(2D)} = \int d^2x \mathcal{L}_{\bar{B}}^{(2D)}$ remain invariant under $s_{(a)d}$.

We close this section with the remark that we also end up with CF-type of restrictions (corresponding to the (anti-)co-BRST symmetry transformations) when s_{ad} and s_d are applied on \mathcal{L}_B and $\mathcal{L}_{\bar{B}}$, respectively. In other words, we have the following [22]

$$\begin{aligned} s_d \mathcal{L}_{\bar{B}} &= \partial_\mu [\mathcal{B} \cdot D^\mu \bar{C} - \varepsilon^{\mu\nu} C \cdot \partial_\nu \bar{C} \times \bar{C}] + i (\partial_\mu A^\mu) \cdot (\mathcal{B} \times \bar{C}), \\ s_{ad} \mathcal{L}_B &= \partial_\mu [\mathcal{B} \cdot D^\mu C + \varepsilon^{\mu\nu} \bar{C} \cdot \partial_\nu C \times C] + i (\partial_\mu A^\mu) \cdot (\mathcal{B} \times C), \end{aligned} \quad (8)$$

which lead to the existence of the following CF-type restrictions:

$$\mathcal{B} \times C = 0, \quad \mathcal{B} \times \bar{C} = 0. \quad (9)$$

These restrictions are (anti-)co-BRST invariant [i.e. $s_{(a)d}(\mathcal{B} \times C) = 0$, $s_{(a)d}(\mathcal{B} \times \bar{C}) = 0$]. Thus, we note that the CF-type restrictions ($\mathcal{B} \times C = 0$, $\mathcal{B} \times \bar{C} = 0$), in the context of (anti-)co-BRST symmetries, are *different* from the CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$ related with the (anti-)BRST symmetry transformations in the sense that:

$$\begin{aligned} s_{ab} [B + \bar{B} + (C \times \bar{C})] &= i [B + \bar{B} + (C \times \bar{C})] \times \bar{C}, \\ s_b [B + \bar{B} + (C \times \bar{C})] &= i [B + \bar{B} + (C \times \bar{C})] \times C. \end{aligned} \quad (10)$$

We note that $s_{(a)d}[\mathcal{B} \times C] = 0$ and $s_{(a)d}[\mathcal{B} \times \bar{C}] = 0$ (under the (anti-)co-BRST symmetry transformations) but this kind of *perfect* symmetry is *not* obeyed by the CF-condition $B + \bar{B} + (C \times \bar{C}) = 0$ in the context of (anti-)BRST symmetries. The latter condition is (anti-)BRST invariant only on the hypersurface where the CF-condition is satisfied. We lay emphasis on the fact that the (anti-)BRST symmetries are *true* for any arbitrary non-Abelian 1-form gauge theory *but* the (anti-)co-BRST symmetries exist *only* for the 2D non-Abelian 1-form gauge theory. Both these symmetries are physically interesting because both are used [21] to prove that the 2D non-Abelian 1-form gauge theory (without any interaction with matter fields) is a *new* model of topological field theory (TFT) which captures a few key properties of Witten-type TFTs and some salient features of Schwarz-type TFTs.

3 Nilpotent (Anti-)BRST Transformations: Horizontality Condition

We very concisely mention here the salient features of the horizontality condition (HC) that leads to the derivation of proper (anti-)BRST symmetry transformations as well as CF-condition ($B + \bar{B} + (C \times \bar{C}) = 0$) within the framework of *usual* superfield approach to BRST formalism. In this context, it is worthwhile to mention that the geometrical strength of the curvature 2-form ($F^{(2)} = dA^{(1)} + i A^{(1)} \wedge A^{(1)} = \frac{(dx^\mu \wedge dx^\nu)}{2!} F_{\mu\nu}$) plays a crucial role in this technique where $F^{(2)}$ is generalized to the suitably chosen supermanifold as:

$$F^{(2)}(x) \rightarrow \tilde{F}^{(2)}(x, \theta, \bar{\theta}) = \left(\frac{dZ^M \wedge dZ^N}{2!} \right) \tilde{F}_{MN}(x, \theta, \bar{\theta}). \quad (11)$$

In the above, $\tilde{F}^{(2)}$ is the supercurvature 2-form which is defined on a $(D, 2)$ -dimensional supermanifold corresponding to a given D -dimensional non-Abelian 1-form gauge theory

where $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + i(\tilde{A}^{(1)} \wedge \tilde{A}^{(1)})$ and $Z^M = (x^\mu, \theta, \bar{\theta})$ are the superspace coordinates. In this expression, we have generalization of the exterior derivative $d = dx^\mu \partial_\mu$ and 1-form connection $A^{(1)} = dx^\mu A_\mu$ (defined on the D -dimensional flat Minkowski space) to $(D, 2)$ -dimensional supermanifold (on which the given D -dimensional gauge theory is generalized). In other words, we have the following:

$$\begin{aligned} d \rightarrow \tilde{d} &= dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, & \tilde{d}^2 &= 0, \\ A^{(1)} &= dx^\mu A_\mu(x) \rightarrow \tilde{A}^{(1)} = dx^\mu B_\mu(x, \theta, \bar{\theta}) + d\theta \bar{F}(x, \theta, \bar{\theta}) + d\bar{\theta} F(x, \theta, \bar{\theta}), \end{aligned} \quad (12)$$

where $[B_\mu(x, \theta, \bar{\theta}), F(x, \theta, \bar{\theta}), \bar{F}(x, \theta, \bar{\theta})]$ are the superfields on the $(D, 2)$ -dimensional supermanifold corresponding to the ordinary fields $[A_\mu(x), C(x), \bar{C}(x)]$ of the D -dimensional non-Abelian gauge theory. The above supermanifolds have the following expansions along the Grassmannian directions of the $(D, 2)$ -dimensional supermanifold [5-7]

$$\begin{aligned} A_\mu(x) &\rightarrow B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i\theta\bar{\theta} S_\mu(x), \\ C(x) &\rightarrow F(x, \theta, \bar{\theta}) = C(x) + i\theta \bar{B}_1 + i\bar{\theta} B_1 + i\theta\bar{\theta} s(x), \\ \bar{C}(x) &\rightarrow \bar{F}(x, \theta, \bar{\theta}) = \bar{C}(x) + i\theta \bar{B}_2 + i\bar{\theta} B_2 + i\theta\bar{\theta} \bar{s}(x), \end{aligned} \quad (13)$$

where, on the r.h.s., we have $(S_\mu, B_1, \bar{B}_1, B_2, \bar{B}_2)$ and $(R_\mu, \bar{R}_\mu, S, \bar{S})$ as the secondary fields which are bosonic and fermionic in nature, respectively. These secondary fields are determined in terms of the basic and auxiliary fields of the theory due to the beauty and strength of HC. We elaborate below some of the key features of HC in a concise manner.

We observe that the kinetic term $(-\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu})$ remains invariant under the nilpotent (anti-)BRST symmetries (because, primarily, it is a gauge invariant quantity). Within the framework of USFA to BRST formalism, we demand that all the Grassmannian components of the (anti-)symmetric supercurvature tensor $\tilde{F}_{MN} = (\tilde{F}_{\mu\theta}, \tilde{F}_{\theta\theta}, \tilde{F}_{\theta\bar{\theta}}, \tilde{F}_{\bar{\theta}\bar{\theta}})$ should be set equal to zero so that we have the following equality of the kinetic term (cf. Eq. (11))

$$-\frac{1}{4} \tilde{F}_{MN}(x, \theta, \bar{\theta}) \cdot \tilde{F}^{MN}(x, \theta, \bar{\theta}) = -\frac{1}{4} F_{\mu\nu}(x) \cdot F^{\mu\nu}(x), \quad (14)$$

which is a gauge invariant restriction (GIR). To achieve the equality (14), one of the simplest choices is to set all the Grassmannian components of $\tilde{F}_{MN}(x, \theta, \bar{\theta})$ equal to zero so that only the antisymmetric spacetime components $\tilde{F}_{\mu\nu}(x, \theta, \bar{\theta})$ survive. To be precise, the restrictions $\tilde{F}_{\mu\theta} = \tilde{F}_{\theta\theta} = \tilde{F}_{\theta\bar{\theta}} = \tilde{F}_{\bar{\theta}\bar{\theta}} = 0$ lead to the following relationship between the secondary fields and basic as well as auxiliary fields (with the identifications $\bar{B}_1 = \bar{B}, B_2 = B$), namely;

$$\begin{aligned} R_\mu &= D_\mu C, & \bar{R}_\mu &= D_\mu \bar{C}, & S_\mu &= (D_\mu B + D_\mu C \times \bar{C}) \equiv -(D_\mu \bar{B} + C \times D_\mu \bar{C}), \\ s &= i(\bar{B} \times C), & \bar{s} &= -i(B \times \bar{C}), & B_1 &= -\frac{1}{2}(C \times C), \\ \bar{B}_2 &= -\frac{1}{2}(\bar{C} \times \bar{C}), & B + \bar{B} &+ (C \times \bar{C}) &= 0, \end{aligned} \quad (15)$$

where the *last* entry is nothing but the celebrated CF-condition [23]. Thus, it is the theoretical strength of HC that we have determined *all* the secondary fields in terms of the basic and auxiliary fields of the theory described by the Lagrangian density (1).

The substitution of the expressions for the above secondary fields into the expansions (13) leads to the following super expansions of the superfields [5-7]

$$\begin{aligned}
B_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(D_\mu \bar{C}) + \bar{\theta}(D_\mu C) + i\theta\bar{\theta}[D_\mu B + D_\mu C \times \bar{C}] \\
&\equiv A_\mu(x) + \theta(s_{ab}A_\mu) + \bar{\theta}(s_b A_\mu) + \theta\bar{\theta}(s_b s_{ab}A_\mu), \\
F^{(h)}(x, \theta, \bar{\theta}) &= C(x) + \theta(i\bar{B}) + \bar{\theta}(-\frac{i}{2}(C \times C)) + \theta\bar{\theta}(-\bar{B} \times C) \\
&\equiv C(x) + \theta(s_{ab}C) + \bar{\theta}(s_b C) + \theta\bar{\theta}(s_b s_{ab}C), \\
\bar{F}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(-\frac{i}{2}(\bar{C} \times \bar{C})) + \bar{\theta}(iB) + \theta\bar{\theta}(B \times \bar{C}) \\
&\equiv \bar{C}(x) + \theta(s_{ab}\bar{C}) + \bar{\theta}(s_b \bar{C}) + \theta\bar{\theta}(s_b s_{ab}\bar{C}),
\end{aligned} \tag{16}$$

where the superscript (h) denotes that the above superfields have been determined after the application of HC. We note that the coefficients of $\theta, \bar{\theta}$ and $\theta\bar{\theta}$ are nothing but the (anti-)BRST symmetries (3) of the D -dimensional non-Abelian gauge theory (without any interactions with matter fields). In other words, we observe that the HC leads to the determination of proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations for the (anti-)ghost and gauge fields of the non-Abelian 1-form gauge theory in any arbitrary dimension of flat Minkowski spacetime.

We end this section with the following important remarks. First of all, it is clear that the kinetic term remains invariant under the (anti-)BRST symmetry transformations. The curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(A_\mu \times A_\nu)$ owes its origin to the exterior derivative ($d = dx^\mu \partial_\mu$) because it is derived from the curvature 2-form $F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}$. Second, we note that the CF-condition is responsible for the existence of the coupled (but equivalent) Lagrangian densities (1) for the non-Abelian 1-form gauge theory. The equivalence can be checked from our observations in Eq. (4). Thus, it is evident that *both* the Lagrangian densities respect the (anti-)BRST symmetry transformations *only* on the hypersurface which is described in the language of CF-condition. Third, we also observe that the absolute anticommutativity property of $s_{(a)b}$ (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) is satisfied if and only if we use the CF-condition (which is one of the hallmarks of a *quantum* gauge theory when it is described within the framework of BRST formalism). Fourth, we note that the CF-condition is an (anti-)BRST invariant (i.e. $s_{(a)b}[B + \bar{B} + (C \times \bar{C})] = 0$) quantity at the *quantum* level (cf. Eq. (10)). Hence, this condition is a *physical* restriction. Fifth, we point out that the surviving component of the super curvature tensor is equal to:

$$\begin{aligned}
\tilde{F}_{\mu\nu}^{(h)}(x, \theta, \bar{\theta}) &= \partial_\mu B_\nu^{(h)} - \partial_\nu B_\mu^{(h)} + i(B_\mu^{(h)} \times B_\nu^{(h)}) \equiv F_{\mu\nu}(x) + \theta(iF_{\mu\nu} \times \bar{C}) \\
&+ \bar{\theta}(iF_{\mu\nu} \times C) + \theta\bar{\theta}[-(F_{\mu\nu} \times C) \times \bar{C} - F_{\mu\nu} \times B].
\end{aligned} \tag{17}$$

The above equation demonstrates that $s_b F_{\mu\nu} = i(F_{\mu\nu} \times C)$, $s_{ab} F_{\mu\nu} = i(F_{\mu\nu} \times \bar{C})$ and $s_b s_{ab} F_{\mu\nu} = -[(F_{\mu\nu} \times B) + (F_{\mu\nu} \times C) \times \bar{C}] \equiv [(F_{\mu\nu} \times \bar{B}) - (F_{\mu\nu} \times \bar{C}) \times C]$. It is self-evident that, for the $2\bar{D}$ non-Abelian theory (where $F_{\mu\nu}$ has only one existing component $F_{01} = E$), we have the following

$$\begin{aligned}
E(x) \rightarrow \tilde{E}^{(h)}(x, \theta, \bar{\theta}) &= E(x) + \theta(iE \times \bar{C}) + \bar{\theta}(iE \times C) \\
&+ \theta\bar{\theta}[-(E \times C) \times \bar{C} - E \times B],
\end{aligned} \tag{18}$$

which implies that $s_b E = i(E \times C)$, $s_{ab} E = i(E \times \bar{C})$, $s_b s_{ab} E = -[(E \times B) + (E \times C) \times \bar{C}]$. As a side remark, we note that this expression would turn out to be useful, later on, in Sec. 5. Sixth, rest of the (anti-)BRST symmetry transformations in (3) are determined due to the requirements of nilpotency and anticommutativity (which are the key properties of (anti-)BRST symmetries). Seventh, it is evident from Eq. (16) that the (anti-)BRST symmetry transformations $s_{(a)b}$ are intimately connected with the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$. Eighth, the nilpotency $(\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0)$ and absolute anticommutativity $(\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0)$ of the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ imply that $s_{(a)b}^2 = 0$ and $s_b s_{ab} + s_{ab} s_b = 0$, too. Finally, we note that $-\frac{1}{4} \tilde{F}_{\mu\nu}^{(h)}(x, \theta, \bar{\theta}) \cdot \tilde{F}^{\mu\nu(h)}(x, \theta, \bar{\theta}) = -\frac{1}{4} F_{\mu\nu}(x) \cdot F^{\mu\nu}(x)$ as desired.

4 (Anti-)co-BRST Symmetries: Superfield Formalism

We derive here the (anti-)co-BRST symmetry transformations (6) by exploiting the ideas of AVSA. In this connection, first of all, we take the generalization of the exterior derivative $d = dx^\mu \partial_\mu$ and ordinary 1-form connection $A^{(1)} = dx^\mu (A_\mu T)$ onto the $(2, 2)$ -dimensional supermanifold as given in (12). We note that the gauge-fixing term of Lagrangian density (1) owes its origin to the co-exterior derivative (δ) , namely;

$$\delta A^{(1)} = - * d * A^{(1)} = \partial_\mu A^\mu, \quad \delta = - * d *, \quad (19)$$

where $*$ is the Hodge duality operation on the $2D$ ordinary flat Minkowski spacetime manifold. One of the salient features of the (anti-)co-BRST symmetry transformations (6) is the observation that it is the gauge-fixing term (owing its origin to the co-exterior derivative $\delta = - * d *$) that remains invariant under them. The relation (19) can be generalized onto our chosen $(2, 2)$ -dimensional supermanifold. Thus, we invoke the following dual-horizontality condition (DHC) (i.e. an analogue of HC), namely;

$$\tilde{\delta} \tilde{A}^{(1)} = \delta A^{(1)}, \quad \tilde{\delta} \tilde{A}^{(1)} = - \star \tilde{d} \star \tilde{A}^{(1)}, \quad (20)$$

where $\tilde{\delta} = - \star \tilde{d} \star$ is the generalization of the ordinary co-exterior derivative $\delta = - * d *$ onto the chosen supermanifold and \star is the Hodge duality operation on the $(2, 2)$ -dimensional supermanifold. For the Abelian 1-form theory, the \star operator has been defined in [24].

In our Appendix A, the step-by-step computation of the l.h.s. of the DHC ($\tilde{\delta} \tilde{A}^{(1)} = \delta A^{(1)}$) has been worked out. We take that result and write it in the following fashion:

$$[\partial_\mu B^\mu + \partial_\theta \bar{F} + \partial_{\bar{\theta}} F] + s^{\bar{\theta}\bar{\theta}} (\partial_{\bar{\theta}} \bar{F}) + s^{\theta\theta} (\partial_\theta F) = \partial_\mu A^\mu. \quad (21)$$

The above equality yields the relationships as listed below:

$$\partial_\theta F = 0, \quad \partial_{\bar{\theta}} \bar{F} = 0, \quad \partial_\mu B^\mu + \partial_\theta \bar{F} + \partial_{\bar{\theta}} F = \partial_\mu A^\mu. \quad (22)$$

This is due to the fact that there is *no* presence of factors like $s^{\theta\theta}$ and $s^{\bar{\theta}\bar{\theta}}$ on the r.h.s. At this stage, we have to take into account the expansion of $B_\mu(x, \theta, \bar{\theta})$, $F(x, \theta, \bar{\theta})$ and $\bar{F}(x, \theta, \bar{\theta})$ along the Grassmannian directions $(\theta, \bar{\theta})$ of the $(2, 2)$ dimensional supermanifold as given in (13). Their substitution leads to the following restrictions from (22), namely;

$$\begin{aligned} \partial_\mu \bar{R}^\mu &= 0, & \partial_\mu R^\mu &= 0, & \partial_\mu S^\mu &= 0, & s &= 0, \\ \bar{B}_1 &= 0, & B_2 &= 0, & \bar{s} &= 0, & B_1 + \bar{B}_2 &= 0, \end{aligned} \quad (23)$$

where $B_1 + \bar{B}_2 = 0$ is the analogue of the CF-type condition. We make the choice $B_1 = -\mathcal{B}$ which implies that $\bar{B}_2 = \mathcal{B}$. Thus, we obtain the expansions of the fermionic superfields $F(x, \theta, \bar{\theta})$ and $\bar{F}(x, \theta, \bar{\theta})$ along the Grassmannian directions $(\theta, \bar{\theta})$ as follows

$$\begin{aligned} F^{(d)}(x, \theta, \bar{\theta}) &= C(x) + \bar{\theta}(-i\mathcal{B}) \equiv C(x) + \bar{\theta}(s_d C), \\ \bar{F}^{(d)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(i\mathcal{B}) \equiv \bar{C}(x) + \theta(s_{ad}\bar{C}), \end{aligned} \quad (24)$$

where the superscript (d) stands for the expansions of the superfields, obtained after the application of the DHC, and $s_{(a)d}$ are the (anti-)co-BRST symmetry transformations for the fields $(\bar{C})C$ that have been quoted in Eq. (6). We note, at this juncture, that we have already derived the (anti-)co-BRST symmetry transformations for the (anti-)ghost fields $(\bar{C})C$ of our theory (cf. Eq. (6)). It is also clear that $\partial_\theta \bar{F}^{(d)} = s_{ad}\bar{C}$ and $\partial_{\bar{\theta}} F^{(d)} = s_d C$. These relationships show that the (anti-)co-BRST symmetry transformations $s_{(a)d}$ can be identified with the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ along the Grassmannian directions $(\theta, \bar{\theta})$, respectively. Moreover, the above identifications imply that $s_{(a)d}^2 = 0$ due to $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$ and $s_d s_{ad} + s_{ad} s_d = 0$ due to $\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0$.

We have to compute now the (anti-)co-BRST symmetry transformations for the gauge field $A_\mu \equiv (A_\mu \cdot T)$. For this purpose, we have to exploit the ideas behind the AVSA where the (anti-)co-BRST invariant quantities would be required to be independent of the “soul” coordinates $(\theta, \bar{\theta})$. In this context, we observe that the following combination of fields (in the square bracket below) is an (anti-)co-BRST invariant quantity, namely;

$$s_{(a)d} \left[\varepsilon^{\mu\nu} A_\nu \cdot \partial_\mu \mathcal{B} - i \partial_\mu \bar{C} \cdot \partial^\mu C \right] = 0. \quad (25)$$

According to the basic tenets of AVSA, we have to equate the quantity in the square bracket with its counterparts in terms of the superfields, namely;

$$\begin{aligned} \varepsilon^{\mu\nu} B_\nu(x, \theta, \bar{\theta}) \cdot \partial_\mu \mathcal{B}(x) - i \partial_\mu \bar{F}^{(d)}(x, \theta, \bar{\theta}) \cdot \partial^\mu F^{(d)}(x, \theta, \bar{\theta}) \\ = \varepsilon^{\mu\nu} A_\nu(x) \cdot \partial_\mu \mathcal{B}(x) - i \partial_\mu \bar{C}(x) \cdot \partial^\mu C(x), \end{aligned} \quad (26)$$

where we have taken $F^{(d)}$ and $\bar{F}^{(d)}$ from Eq. (24) and $\mathcal{B}(x) \rightarrow \mathcal{B}(x, \theta, \bar{\theta}) = \mathcal{B}(x)$ because of the fact that $s_{(a)d} \mathcal{B}(x) = 0$. Hence, it will have *no* expansion along $(\theta, \bar{\theta})$ -directions of the $(2, 2)$ -dimensional supermanifold provided we accept the result that the coefficients of θ and $\bar{\theta}$ correspond to the (anti-)co-BRST symmetry transformations, respectively. The above restriction (cf. Eq. (26)) is called as the dual-gauge invariant restriction (DGIR). Physically, this restriction implies that the (anti-)co-BRST invariant quantities should remain independent of the “soul” coordinates $(\theta, \bar{\theta})$ because the latter are *only* the mathematical artifacts. It will be noted that we have taken the expansions from (24) for the generalization of the fields $C(x)$ and $\bar{C}(x)$. The explicit substitutions, from (24) into (26), yield:

$$\varepsilon^{\mu\nu} \bar{R}_\nu + \partial^\mu C = 0, \quad \varepsilon^{\mu\nu} R_\nu + \partial^\mu \bar{C} = 0, \quad \varepsilon^{\mu\nu} S_\nu - \partial^\mu \mathcal{B} = 0. \quad (27)$$

The above relations lead to the derivation of the secondary fields of the expansions (that is present for $B_\mu(x, \theta, \bar{\theta})$ in (13)) as:

$$R_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad \bar{R}_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, \quad S_\mu = \varepsilon_{\mu\nu} \partial^\nu \mathcal{B}. \quad (28)$$

Thus, ultimately, we obtain the expansions of the superfield $B_\mu(x, \theta, \bar{\theta})$ as

$$\begin{aligned} B_\mu^{(dg)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(-\varepsilon_{\mu\nu}\partial^\nu C) + \bar{\theta}(-\varepsilon_{\mu\nu}\partial^\nu \bar{C}) + \theta\bar{\theta}(i\varepsilon_{\mu\nu}\partial^\nu \mathcal{B}) \\ &\equiv A_\mu(x) + \theta(s_{ad}A_\mu) + \bar{\theta}(s_dA_\mu) + \theta\bar{\theta}(s_d s_{ad}A_\mu), \end{aligned} \quad (29)$$

where the superscript (dg) denotes that the above expansion has been obtained after the application of the dual-gauge invariant restriction (DGIR). The above expansion leads to the derivations of (anti-)co-BRST symmetry transformations for the gauge field $A_\mu \equiv (A_\mu \cdot T)$ (cf. Eq. (6)) as the coefficients of the Grassmanian variables θ and $\bar{\theta}$. The noteworthy point, at this juncture, is the fact that DHC and DGIR are intertwined *together* in a very useful fashion in Eq. (26). This is the beauty and strength of AVSA.

We note, from Eq. (6), that the component $F_{01} = E = -\varepsilon^{\mu\nu}(\partial_\mu A_\nu + \frac{i}{2}A_\mu \times A_\nu)$ of the curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(A_\mu \times A_\nu)$ transforms under the (anti-)co-BRST symmetry transformations as: $s_d E = D_\mu \partial^\mu \bar{C}$ and $s_{ad} E = D_\mu \partial^\mu C$. These transformations can be derived from the AVSA to BRST formalism as:

$$\begin{aligned} E \rightarrow \tilde{E}(x, \theta, \bar{\theta}) &= -\varepsilon^{\mu\nu} \left[\partial_\mu B_\nu^{(dg)} + \frac{i}{2} (B_\mu^{(dg)} \times B_\nu^{(dg)}) \right] \\ &\equiv E(x) + \theta(D_\mu \partial^\mu C) + \bar{\theta}(D_\mu \partial^\mu \bar{C}) \\ &\quad + \theta\bar{\theta}[-i D_\mu \partial^\mu \mathcal{B} - i\varepsilon_{\mu\nu} \partial^\nu \bar{C} \times \partial^\mu C]. \end{aligned} \quad (30)$$

We observe that the coefficients of $\theta, \bar{\theta}$ and $\theta\bar{\theta}$ do lead to the derivation of $s_{ad}E$, $s_d E$ and $s_d s_{ad}E$. In other words, we obtain $\partial_{\bar{\theta}} \tilde{E}(x, \theta, \bar{\theta})|_{\bar{\theta}=0} = s_{ad} E$ and $\partial_{\theta} \tilde{E}(x, \theta, \bar{\theta})|_{\theta=0} = s_d E$ which imply that the translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ correspond to the (anti-)co-BRST symmetry transformations $(s_{(a)d})$. Thus, we note that $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$ and $\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0$ are intimately connected with the nilpotency (i.e. $s_{(a)d}^2 = 0$) and absolute anticommutativity (i.e. $s_d s_{ad} + s_{ad} s_d = 0$) properties of the (anti-)co-BRST symmetry transformations $(s_{(a)d})$.

5 CF-Type Restrictions: Superfield Approach

We discuss here the derivation of *all* possible CF-type restrictions that could be, in general, supported by the 2D non-Abelian theory within the framework of AVSA. First of all, we observe *here* that these physically motivated restrictions have been derived in our earlier work [22] by exploiting the idea of (anti-)BRST and (anti-)co-BRST symmetry invariance. Thus, our central goal is to derive them by demanding, first of all, that the original CF-condition $[B + \bar{B} + (C \times \bar{C}) = 0]$ should be invariant under the (anti-)co-BRST symmetry transformations. In the language of AVSA, we demand that this condition should be valid on the $(2, 2)$ -dimensional supermanifold, too, namely;

$$B(x) + \bar{B}(x) + [F^{(d)}(x, \theta, \bar{\theta}) \times \bar{F}^{(d)}(x, \theta, \bar{\theta})] = B(x) + \bar{B}(x) + [C(x) \times \bar{C}(x)], \quad (31)$$

where we have taken the generalizations: $B(x) \rightarrow \tilde{B}(x, \theta, \bar{\theta}) = B(x)$, $\bar{B}(x) \rightarrow \tilde{\bar{B}}(x, \theta, \bar{\theta}) = \bar{B}(x)$ due to the fact that *both* these auxiliary fields are (anti-)co-BRST invariant quantities (i.e. $s_{(a)d}B(x) = 0$, $s_{(a)d}\bar{B}(x) = 0$). In other words, the superfields $\tilde{B}(x, \theta, \bar{\theta})$ and $\tilde{\bar{B}}(x, \theta, \bar{\theta})$

have *no* expansions along θ and $\bar{\theta}$ directions of the $(2, 2)$ -dimensional supermanifold. Plugging in the expansions from Eq. (24), we obtain (from the above) the following

$$i\theta(\mathcal{B} \times C) + i\theta(\mathcal{B} \times \bar{C}) + \theta\bar{\theta}(\mathcal{B} \times \mathcal{B}) = 0, \quad (32)$$

which leads to the CF-type restrictions: $\mathcal{B} \times C = 0$, $\mathcal{B} \times \bar{C} = 0$ because $\mathcal{B} \times \mathcal{B} = 0$ automatically. We observe that these CF-type restrictions have appeared earlier in Eqs. (8) and (9) in the Lagrangian formulation. We further observe, at this stage, that these *new* CF-type restrictions are invariant under the nilpotent (anti-)co-BRST symmetry transformations $s_{(a)d}$ [i.e. $s_{(a)d}(\mathcal{B} \times C) = 0$, $s_{(a)d}(\mathcal{B} \times \bar{C}) = 0$]. However, these are *not* invariant under the (anti-)BRST symmetry transformations. To have (anti-)BRST and (anti-)co-BRST symmetries *together* in the $2D$ theory, we have to demand that these *new* CF-type restrictions ($\mathcal{B} \times C = 0$, $\mathcal{B} \times \bar{C} = 0$) should *also* remain invariant under the (anti-)BRST symmetry transformations. Thus, within the framework of AVSA to BRST formalism, we demand the following equalities, namely;

$$\begin{aligned} \tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta}) \times F^{(h)}(x, \theta, \bar{\theta}) &= \mathcal{B}(x) \times C(x), \\ \tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta}) \times \bar{F}^{(h)}(x, \theta, \bar{\theta}) &= \mathcal{B}(x) \times \bar{C}(x), \end{aligned} \quad (33)$$

where the expansions for $F^{(h)}(x, \theta, \bar{\theta})$ and $\bar{F}^{(h)}(x, \theta, \bar{\theta})$ are given in Eq. (16) which have been obtained after the application of HC. The expansions for $\tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta})$ along the Grassmannian directions $(\theta, \bar{\theta})$ can be written as

$$\tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta}) = \mathcal{B} + \theta[i(\mathcal{B} \times \bar{C})] + \bar{\theta}[i(\mathcal{B} \times C)] + \theta\bar{\theta}[-(\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}], \quad (34)$$

in view of the (anti-)BRST symmetry transformations: $s_b \mathcal{B} = i(\mathcal{B} \times C)$, $s_{ab} \mathcal{B} = i(\mathcal{B} \times \bar{C})$ and $s_b s_{ab} \mathcal{B} = [-(\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}]$ on the auxiliary field $\mathcal{B}(x)$. The superscript (g) on the superfield $\mathcal{B}(x, \theta, \bar{\theta})$ has been taken into account to denote that the above expansion, for this superfield, has been derived due to the GIR which we explain below.

We exploit the theoretical strength of AVSA to BRST formalism to determine the super expansion for the superfield $\mathcal{B}(x, \theta, \bar{\theta})$. In this connection, we note that $s_{(a)b}[\mathcal{B} \cdot E] = 0$. Thus, the basic tenets of AVSA permits us to demand the following equality

$$\mathcal{B}(x, \theta, \bar{\theta}) \cdot \tilde{E}^{(h)}(x, \theta, \bar{\theta}) = \mathcal{B}(x) \cdot E(x), \quad (35)$$

where the explicit expansion of $\tilde{E}^{(h)}(x, \theta, \bar{\theta})$ has been quoted in Eq. (18). The substitution of this result into the above equation implies that we have the following

$$P(x) = i(\mathcal{B} \times \bar{C}), \quad Q(x) = i(\mathcal{B} \times C), \quad M(x) = i[(\mathcal{B} \times B) + (\mathcal{B} \times C) \times \bar{C}], \quad (36)$$

where the fields $P(x)$, $Q(x)$ and $M(x)$, in the above, are the secondary fields in the general super expansions of $\mathcal{B}(x, \theta, \bar{\theta})$ as given below:

$$\mathcal{B}(x) \rightarrow \tilde{\mathcal{B}}(x, \theta, \bar{\theta}) = \mathcal{B}(x) + \theta P(x) + \bar{\theta} Q(x) + i\theta\bar{\theta} M(x). \quad (37)$$

The results in (36) show that $P(x)$ and $Q(x)$ are fermionic in nature in contrast to the bosonic nature of $M(x)$. This observation is *also* consistent with the fermionic nature of the Grassmannian variables $\theta, \bar{\theta}$ that are present on the r.h.s. of the expansion (37).

We focus now on the explicit form of the restrictions (33). These can be expanded as:

$$\begin{aligned} & \left(\mathcal{B} + \theta [i (\mathcal{B} \times \bar{C})] + \bar{\theta} [i (\mathcal{B} \times C)] + \theta \bar{\theta} [- (\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}] \right) \times \\ & \left(C + \theta (i \bar{B}) + \bar{\theta} \left[\frac{-i}{2} (\bar{C} \times C) \right] + \theta \bar{\theta} [-\bar{B} \times C] \right) = \mathcal{B}(x) \times C(x), \end{aligned} \quad (38)$$

$$\begin{aligned} & \left(\mathcal{B} + \theta [i (\mathcal{B} \times \bar{C})] + \bar{\theta} [i (\mathcal{B} \times C)] + \theta \bar{\theta} [- (\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}] \right) \times \\ & \left(\bar{C} + \theta \left[\frac{-i}{2} (\bar{C} \times \bar{C}) \right] + \bar{\theta} [i B] + \theta \bar{\theta} [B \times C] \right) = \mathcal{B}(x) \times \bar{C}(x). \end{aligned} \quad (39)$$

Setting the coefficients of $\theta, \bar{\theta}$ and $\theta \bar{\theta}$ equal to zero (in the above), we obtain the following restrictions $\mathcal{B} \times \bar{B} = 0$, $\mathcal{B} \times B = 0$ where we have used $\mathcal{B} \times C = 0$, $\mathcal{B} \times \bar{C} = 0$ and $B + \bar{B} + (C \times \bar{C}) = 0$ which are the *original* restrictions on the theory. To be more precise, these restrictions have already appeared earlier in the Lagrangian formulation [22].

We note that the CF-type restriction $\mathcal{B} \times \bar{B} = 0$ and $\mathcal{B} \times B = 0$ are, once again, invariant under the (anti-)co-BRST symmetry transformations (i.e. $s_{(a)d} [\mathcal{B} \times B] = 0$, $s_{(a)d} [\mathcal{B} \times \bar{B}] = 0$). Thus, we demand their invariance under the (anti-)BRST symmetries (in view of having *both* the (anti-)BRST and (anti-)co-BRST symmetries together in the 2D theory) with the following restrictions

$$\begin{aligned} \tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta}) \times \tilde{B}^{(g)}(x, \theta, \bar{\theta}) &= \mathcal{B}(x) \times B(x), \\ \tilde{\mathcal{B}}^{(g)}(x, \theta, \bar{\theta}) \times \tilde{\bar{B}}^{(g)}(x, \theta, \bar{\theta}) &= \mathcal{B}(x) \times \bar{B}(x), \end{aligned} \quad (40)$$

where, the generalizations and super expressions of $\mathcal{B}(x), B(x)$ and $\bar{B}(x)$ fields, onto the (2, 2)-dimensional supermanifold are (34) and the following:

$$\begin{aligned} B(x) &\rightarrow \tilde{B}^{(g)}(x, \theta, \bar{\theta}) = B(x) + \theta (i [B \times \bar{C}]) + \bar{\theta} (0) + \theta \bar{\theta} (0), \\ \bar{B}(x) &\rightarrow \tilde{\bar{B}}^{(g)}(x, \theta, \bar{\theta}) = \bar{B}(x) + \theta (0) + \bar{\theta} (i [\bar{B} \times C]) + \theta \bar{\theta} (0). \end{aligned} \quad (41)$$

In the above, the superscript (g) denotes the fact that the superfields $\tilde{B}^{(g)}(x, \theta, \bar{\theta})$ and $\tilde{\bar{B}}^{(g)}(x, \theta, \bar{\theta})$ corresponds to the superfields that could be obtained after the application of GIRs. We elaborate the derivation of these within the framework of AVSA. In this context, we note that $s_b B(x) = 0$ and $s_b (B \times \bar{C}) = 0$. Thus, we have the following GIRs

$$\partial_{\bar{\theta}} \tilde{B}(x, \theta, \bar{\theta}) = 0, \quad \partial_{\bar{\theta}} [\tilde{B}(x, \theta, \bar{\theta}) \times \bar{F}^{(h)}(x, \theta, \bar{\theta})] = 0, \quad (42)$$

where $\bar{F}^{(h)}(x, \theta, \bar{\theta})$ has been expressed in Eq. (16) and the general super expansion of $\tilde{B}(x, \theta, \bar{\theta})$ along the Grassmannian directions $(\theta, \bar{\theta})$ is as follows:

$$B(x) \rightarrow \tilde{B}(x, \theta, \bar{\theta}) = B(x) + \theta U(x) + \bar{\theta} V(x) + \theta \bar{\theta} S(x). \quad (43)$$

Here the secondary fields $(U(x), V(x))$ are fermionic and $S(x)$ is a bosonic secondary field due to the fermionic nature of $(\theta, \bar{\theta})$ and bosonic nature of $B(x)$. We have also taken into account the mapping $\partial_{\bar{\theta}} \leftrightarrow s_b$. It will be noted that $\partial_{\bar{\theta}} \tilde{B}(x, \theta, \bar{\theta}) = 0$ implies that

$V(x) = S(x) = 0$. Thus, the reduced form of $\tilde{B}(x, \theta, \bar{\theta})$ is $\tilde{B}^{(r)}(x, \theta, \bar{\theta}) = B(x) + \theta U(x)$. Now the *second* restriction in (42) can be expressed as:

$$\begin{aligned} \partial_{\bar{\theta}} [\tilde{B}^{(r)}(x, \theta, \bar{\theta}) \times F^{(h)}(x, \theta, \bar{\theta})] = \\ \frac{\partial}{\partial \theta} [\{B(x) + \theta U(x)\} \times \{\bar{C} + \theta \left(\frac{-i}{2} (\bar{C} \times \bar{C})\right) + \bar{\theta} (iB) + \theta \bar{\theta} (B \times \bar{C})\}]. \end{aligned} \quad (44)$$

The above restriction produces the form of $\tilde{B}(x, \theta, \bar{\theta})$ that has been quoted in Eq. (41) as $\tilde{B}^{(g)}(x, \theta, \bar{\theta})$. In an exactly similar fashion, we observe that $s_{ab} \bar{B} = 0$ and $s_{ab} (\bar{B} \times C) = 0$. Thus, we have the following restrictions, within the framework of AVSA, on supermanifolds:

$$\partial_{\theta} [\tilde{B}(x, \theta, \bar{\theta})] = 0, \quad \partial_{\theta} [\tilde{B}(x, \theta, \bar{\theta}) \times F^{(h)}(x, \theta, \bar{\theta})] = 0, \quad (45)$$

where the general expansion for the superfield $\tilde{B}(x, \theta, \bar{\theta})$ is

$$\bar{B}(x) \rightarrow \tilde{B}(x, \theta, \bar{\theta}) = \bar{B}(x) + \theta K(x) + \bar{\theta} L(x) + i \theta \bar{\theta} N(x). \quad (46)$$

In the above, $(K(x), L(x))$ are the fermionic secondary fields, $N(x)$ is bosonic and we have taken into account $\partial_{\theta} \leftrightarrow s_{ab}$. The first condition in (45) leads to

$$K(x) = 0, \quad N(x) = 0, \quad \tilde{B}(x, \theta, \bar{\theta}) \rightarrow \tilde{B}^{(r)}(x, \theta, \bar{\theta}) = \bar{B}(x) + \bar{\theta} L(x), \quad (47)$$

where the superscript (r) denotes the reduced form of $\tilde{B}(x, \theta, \bar{\theta})$. Now plugging in the values of $F^{(h)}(x, \theta, \bar{\theta})$ from Eq. (16) and $\tilde{B}^{(r)}(x, \theta, \bar{\theta})$, we obtain (from (45)) the following:

$$\frac{\partial}{\partial \theta} \left[(\bar{B}(x) + \bar{\theta} L(x)) \times (C(x) + \theta (i \bar{\theta}) + \bar{\theta} [-\frac{i}{2} (C \times C)] + \theta \bar{\theta} (-\bar{B} \times C)) \right] = 0, \quad (48)$$

which, ultimately, yields the values of $L(x) = i(\bar{B} \times C)$. Substitution of this value in $\tilde{B}^{(r)}(x, \theta, \bar{\theta})$ produces $\tilde{B}^{(g)}(x, \theta, \bar{\theta})$ which has been quoted in Eq. (41). The above expansions agree with the (anti-)BRST symmetry transformations $s_b B = 0$, $s_{ab} B = i(B \times \bar{C})$, $s_b \bar{B} = i(\bar{B} \times C)$, $s_{ab} \bar{B} = 0$ which can also be derived by the requirements of the nilpotency and absolute anticommutativity properties.

The explicit substitutions, from (34) and (41) into (40), imply the following equalities:

$$\begin{aligned} (\mathcal{B} + \theta [i(\mathcal{B} \times \bar{C})] + \bar{\theta} [i(\mathcal{B} \times C) + \theta \bar{\theta} [-(\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}]] \times \\ (\mathcal{B} + i \theta (\mathcal{B} \times \bar{C})) = \mathcal{B}(x) \times B(x), \end{aligned} \quad (49)$$

$$\begin{aligned} (\mathcal{B} + \theta [i(\mathcal{B} \times \bar{C})] + \bar{\theta} [i(\mathcal{B} \times C) + \theta \bar{\theta} [-(\mathcal{B} \times B) - (\mathcal{B} \times C) \times \bar{C}]] \times \\ (\bar{B} + i \bar{\theta} (\bar{B} \times C)) = \mathcal{B}(x) \times \bar{B}(x). \end{aligned} \quad (50)$$

The above equalities lead, ultimately, to the following *new* restrictions:

$$\mathcal{B} \times B = 0, \quad \bar{B} \times C = 0, \quad B \times \bar{B} = 0, \quad B \times \bar{C} = 0. \quad (51)$$

It is evident that the above restrictions are neither *perfectly* invariant under the (anti-)co-BRST symmetries *nor* under the (anti-)BRST symmetry transformations. The substitutions of the superfields $F^{(d)}$, $\bar{F}^{(d)}$, $F^{(h)}$, $\bar{F}^{(h)}$, $\tilde{B}^{(g)}$, $\tilde{\bar{B}}^{(g)}$, $\mathcal{B}^{(g)}$ in a straightforward manner (from appropriate equations) lead to the derivations of the new restrictions:

$$B \times C = 0, \quad \bar{B} \times \bar{C} = 0. \quad (52)$$

At this stage, the tower of restrictions terminate and there are *no* further CF-type restrictions on the $2D$ non-Abelian 1-form gauge theory. Thus, we have derived here *all* possible CF-type restrictions that could be supported by the self-interacting $2D$ non-Abelian theory (without any interaction with matter fields).

6 Conclusions

In our present endeavor, we have applied the geometrical AVSA to BRST formalism for the derivation of proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST and (anti-)co-BRST transformations for the self-interacting $2D$ non-Abelian 1-form gauge theory (without any interaction with matter fields). This exercise has been specifically performed in the case of $2D$ non-Abelian 1-form gauge theory where the (anti-)BRST and (anti-)co-BRST symmetries co-exist *together* (see, e.g. [21]). In fact, these nilpotent symmetries prove that this $2D$ non-Abelian model is a tractable physical example of the Hodge theory as well as a *new* model of topological field theory (see, e.g. [21]). The latter claim is true because it is corroborated by the observation that the $2D$ non-Abelian theory captures a few key properties of the Witten-type TFT and some salient features of the Schwarz-type TFT. The decisive features of the above continuous symmetries (and their (anti)commutators) is the observation that these *symmetries* (and their corresponding *conserved charges*) provide the physical realizations of the de Rham cohomological operators of differential geometry. Hence, our present $2D$ self-interacting non-Abelian field theoretic model turns out to be an example of the Hodge theory.

In our present investigation, we have applied the DHC and DGIR to obtain the (anti-)co-BRST symmetry transformations for the $2D$ non-Abelian 1-form gauge theory within the framework of AVSA to BRST formalism. This result is completely *novel* as, in our previous attempts [10-14], we have *not* applied the AVSA to derive the (anti-)co-BRST symmetry transformations systematically for the $2D$ non-Abelian theory. Further, we have exploited the theoretical potential and power of the AVSA to BRST formalism to derive *all* possible CF-type restrictions that could emerge from the *original* CF-condition ($B + \bar{B} + (C \times \bar{C}) = 0$) by demanding its invariance under the (anti-)BRST and (anti-)co-BRST symmetries. Of course, the latter requirements (i.e. (anti-)BRST and (anti-)co-BRST invariances) have been expressed in the language of appropriately chosen superfields (that have been derived after the application of (D)HCs and (D)GIRs). The upshot of this whole exercise is the emergence of a tower of CF-type restrictions that could be, in general, supported by the $2D$ non-Abelian theory. It should be pointed out, however, that only a few of these CF-type restrictions have been *actually* utilized by us in the Lagrangian formulation where the (anti-)BRST and (anti-)co-BRST symmetry transformations have been discussed.

In our earlier works (see, e.g. [15-17]), we have claimed that the (anti-)dual-BRST symmetry can exist for the p -form ($p = 1, 2, 3, \dots$) gauge theories *only* in the $2p$ -dimensions of spacetime. Thus, for the (non-)Abelian 1-form gauge theories, the above (anti-)co-BRST symmetries exist *only* in two (1+1)-dimensions of spacetime. We have also demonstrated the existence of (anti-)co-BRST symmetries in the cases of $4D$ Abelian 2-form and $6D$ Abelian 3-form gauge theories which corroborates the claims that have been made in our earlier work [16]. It would be a nice future endeavor for us to apply geometrical AVSA to BRST formalism for such theoretically interesting systems to obtain the (anti-)co-BRST symmetries and tower of all CF-type restrictions. We are busy, at the moment, in exploring the proof of the above speculative ideas and our results would be reported elsewhere.

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Appendix A: On the Derivation of $\tilde{\delta}\tilde{A}^{(1)} = -\star\tilde{d}\star\tilde{A}^{(1)}$

We carry out here the step-by-step computation of the l.h.s. of the DHC (i.e. $\tilde{\delta}\tilde{A}^{(1)} = \delta A^{(1)}$) by applying the Hodge duality \star operation [24] on the (2, 2)-dimensional supermanifold that has been chosen for our discussions. First of all, we derive the following 3-form, namely;

$$\begin{aligned}\star\tilde{A}^{(1)} &= \star\left[dx^\mu B_\mu + d\theta\bar{F} + d\bar{\theta}F\right] = \varepsilon^{\mu\nu}(dx_\nu \wedge d\theta \wedge d\bar{\theta})B_\mu \\ &+ \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\bar{\theta})\bar{F} + \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\theta)F,\end{aligned}\quad (53)$$

where we have used the following duality operations [24]

$$\begin{aligned}\star(dx^\mu) &= \varepsilon^{\mu\nu}(dx_\nu \wedge d\theta \wedge d\bar{\theta}), & \star(d\theta) &= \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\bar{\theta}), \\ \star(d\bar{\theta}) &= \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\theta),\end{aligned}\quad (54)$$

because of the fact that the Hodge dual of a 1-form, on a (2, 2)-dimensional supermanifold, is a 3-form. Now we apply the super exterior derivative \tilde{d} on (53) to obtain a 4-form on the (2, 2)-dimensional supermanifold as:

$$\begin{aligned}\tilde{d}\star[\tilde{A}^{(1)}] &= \varepsilon^{\mu\nu}(dx_\lambda \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta})\partial^\lambda B_\mu + \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\lambda \wedge dx_\mu \wedge dx_\nu \wedge d\bar{\theta})\partial^\lambda \bar{F} \\ &+ \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\lambda \wedge dx_\mu \wedge dx_\nu \wedge d\theta)\partial^\lambda F - \varepsilon^{\mu\nu}(dx_\nu \wedge d\theta \wedge d\bar{\theta} \wedge d\bar{\theta})\partial_\theta B_\mu \\ &- \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta})\partial_\theta \bar{F} - \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\theta)\partial_\theta F \\ &- \varepsilon^{\mu\nu}(dx_\nu \wedge d\bar{\theta} \wedge d\theta \wedge d\bar{\theta})\partial_{\bar{\theta}} B_\mu - \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\bar{\theta} \wedge d\bar{\theta})\partial_{\bar{\theta}} \bar{F} \\ &- \frac{1}{2!}\varepsilon^{\mu\nu}(dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta})\partial_{\bar{\theta}} F,\end{aligned}\quad (55)$$

where we have used the explicit expression of $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$ and have taken into account the anticommutativity property of the Grassmannian variables $(\theta, \bar{\theta})$ with their derivatives $(\partial_\theta, \partial_{\bar{\theta}})$ and the same property among themselves.

We are in the position now to apply an star $(-\star)$ on the above 4-form to get a scalar (i.e. 0-form). Before we apply it, we would like to state that the *second* and *third* terms of (55) would be equal to *zero* because a 3-form in spacetime differentials (i.e. $dx_\lambda \wedge dx_\mu \wedge dx_\nu$) can *not* exist on a $(2, 2)$ -dimensional supermanifold. Further, as per the rules of the Hodge duality \star operation laid down in our earlier work [24], we can *not* have the existence of 3-form differentials (e.g. $d\theta \wedge d\theta \wedge d\theta$, $d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta}$, $d\theta \wedge d\bar{\theta} \wedge d\bar{\theta}$, $d\theta \wedge d\theta \wedge d\bar{\theta}$) on the $(2, 2)$ -dimensional supermanifold as it can accommodate *only* 2-form differentials in the Grassmannian variables (e.g. $d\theta \wedge d\theta$, $d\bar{\theta} \wedge d\bar{\theta}$, $d\theta \wedge d\bar{\theta}$). As a consequence, the *fourth* and *seventh* terms would be zero. Thus, the existing terms are:

$$\begin{aligned} \tilde{d} \star [\tilde{A}^{(1)}] &= \varepsilon^{\mu\nu} (dx_\lambda \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) \partial^\lambda B_\mu - \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) \partial_\theta \bar{F} \\ &- \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\theta) \partial_\theta F - \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) \partial_{\bar{\theta}} F \\ &- \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\bar{\theta} \wedge d\bar{\theta}) \partial_{\bar{\theta}} \bar{F}. \end{aligned} \quad (56)$$

Now, the application of $(-\star)$ on the above equation leads to the derivation of Eq. (21) (i.e. $\tilde{\delta}\tilde{A}^{(1)} = -\star \tilde{d} \star \tilde{A}^{(1)}$) that has been incorporated in our text. In this derivation, the following inputs have been used (see, e.g. [24] for details)

$$\begin{aligned} \star (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon_{\mu\nu}, \\ \star (dx_\mu \wedge dx_\nu \wedge d\bar{\theta} \wedge d\bar{\theta}) &= \varepsilon_{\mu\nu} s^{\bar{\theta}\bar{\theta}}, \\ \star (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\theta) &= \varepsilon_{\mu\nu} s^{\theta\theta}, \end{aligned} \quad (57)$$

where $s^{\theta\theta}$ and $s^{\bar{\theta}\bar{\theta}}$ are the factors that have been taken into account so that another (\star) operation on (57) yields the original 4-forms with factor of \pm signs in front of them [24].

References

- [1] See, e.g., C. N. Yang, *Physics Today* **33**, 42 (1980)
- [2] J. Thierry-Mieg, *J. Math. Phys.* **21**, 2834 (1980)
- [3] J. Thierry-Mieg, *Nuovo Cimento A* **56**, 396 (1980)
- [4] M. Quiros, F. J. de Urries, J. Hoyos, M. L. Mazon, E. Rodrigues, *J. Math. Phys.* **22**, 1767 (1981)
- [5] L. Bonora, M. Tonin, *Phys. Lett. B* **98**, 48 (1981)
- [6] L. Bonora, P. Pasti, M. Tonin, *Nuovo Cimento A* **63**, 353 (1981)
- [7] L. Bonora, P. Cotta-Ramusino, *Commun. Math. Phys.* **87**, 589 (1983)

- [8] R. Delbourgo, P. D. Jarvis, G. Thompson, *Phys. Lett. B* **109**, 25 (1982)
- [9] L. Baulieu, J. Thierry-Mieg, *Nucl. Phys. B* **197**, 477 (1982)
- [10] See, e.g., R. P. Malik, *Eur. Phys. J. C* **60**, 457 (2009)
- [11] See, e.g., R. P. Malik, *J. Phys. A: Math. Theor.* **39**, 10575 (2006)
- [12] See, e.g., R. P. Malik, *Eur. Phys. J. C* **51**, 169 (2007)
- [13] See, e.g., R. P. Malik, *J. Phys. A* **37**, 5261 (2004)
- [14] See, e.g., R. P. Malik, *Eur. Phys. J. C* **48**, 825 (2006)
- [15] See, e.g., S. Gupta, R. P. Malik, *Eur. Phys. J. C* **58**, 517 (2008)
- [16] R. Kumar, S. Krishna, A. Shukla, R. P. Malik,
Int. J. Mod. Phys. A **29**, 1450135 (2014)
- [17] See, e.g., R. P. Malik, *Eur. Phys. J. C* **60**, 457 (2009)
- [18] K. Nishijima, *Prog. Theor. Phys.* **80**, 905 (1988)
- [19] D. Dudal, V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, M. Picariello,
JHEP **0212**, 008 (2002)
- [20] D. Dudal, H. Verschelde, V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, M. Picariello,
A. Vicini, J. A. Gracey, *JHEP* **0306**, 003 (2003)
- [21] See, e.g., R. P. Malik, *J. Phys. A* **34**, 4167 (2001)
- [22] N. Srinivas, S. Kumar, B. K. Kureel, R. P. Malik, arXiv: 1606.05870 [hep-th]
- [23] G. Curci, R. Ferrari, *Phys. Lett. B* **63**, 91 (1976)
- [24] See, e.g., R. P. Malik, *Int. J. Mod. Phys. A* **21**, 3307 (2006)